Bosonic closed string theory in flat space

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau \, \sqrt{-\gamma}\, \gamma^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$\alpha' = \ell^2/2$ $\quad \ell$ - string length

symmetry (local) $\quad$ reparametrization invariances

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$

$$\delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}$$

$$\delta(\sqrt{h}) = \partial_\alpha(\xi^\alpha \sqrt{h})$$

Weyl scaling $\quad \delta h^{\alpha\beta} = \Lambda h^{\alpha\beta}$

Global symmetry $\quad$ Poincare invariance

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \qquad a_{\mu\nu} = -a_{\nu\mu}$$

$$\delta h^{\alpha\beta} = 0$$

index raised or lowered by $\eta_{\alpha\beta}$

Using repara invariances we set $\gamma_{ab} = \eta_{ab}$

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau \, \partial_a X^\mu \partial^a X_\mu$$

← string with a global symmetry

Static gauge $X^1 = \sigma \quad X^0 = \tau$

a string moving in d-dim flat space

$$= \frac{1}{2\pi\alpha'}\int d^2\sigma \, \partial_+ X^\mu \partial_- X_\mu$$

$\sigma^+ = \tau - \sigma$

$\sigma^- = \tau + \sigma$

eq of motion for X^μ $\quad \partial_+\partial_- X^\mu = 0$ (free wave equation)

With the periodic boundary condition

$$X^\mu(\sigma) = X^\mu(\sigma + 2\pi)$$

$$X^\mu = x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n}\left(\alpha^\mu_n e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}\right)$$

↑ center of mass momentum

Impose the canonical commutation relation

$$[\dot{X}^\mu(\sigma,\tau), \; X^\nu(\sigma',\tau)] = 2\pi\alpha' i \eta^{\mu\nu} \delta(\sigma-\sigma')$$

$$\Rightarrow \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}$$

$$[\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0} \quad \sim \text{harmonic oscillator}$$

$$(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n} \qquad (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$$

energy momentum tensor $\quad T_{ab} \sim \dfrac{\delta S}{\delta \gamma_{ab}}$

(eq of motion $T_{ab} = 0$)

Conformal invariance $\to$ $\gamma^{ab} T_{ab} = 0$

$T_{+-} = 0$ automatically satisfied

$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + (X')^2)$

$$T_{++} = \frac{1}{\alpha'}(\partial_+ X^\mu)(\partial_+ X_\mu)$$

$$= \sum_n L_n e^{-in(\tau-\sigma)}$$

$$T_{--} = \frac{1}{\alpha'}(\partial_- X^\mu)(\partial_- X_\mu) = \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}$$

where

$$L_n = \sum_m \frac{1}{2} \alpha_{n-m} \cdot \alpha_m \qquad \tilde{L}_n = \sum_m \frac{1}{2} \tilde{\alpha}_{n-m} \tilde{\alpha}_m$$

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu$$

Hamiltonian $H = L_0 + \tilde{L}_0$

diffeomorphism invariance $\to$ constraints $T_{++} = T_{--} = 0$

as a quantum operator L_n, $\tilde{L}_n$ are well defined

while L_0 and $\tilde{L}_0$ are subject to the normal ordering ambiguity.

Oscillator ground state $\alpha_m |0\rangle = 0$

one can have $|0 ; p^\mu\rangle$

One problem is that for time component

$[\alpha^0_n, \alpha^{0\dagger}_n] = -n$ so we have negative norm states

e.g. $\langle 0| \alpha^0_n \alpha^{0\dagger}_n |0\rangle = -n$

We need additional restrictions for the Hilbert space to obtain a sensible theory.

For a physical state, we impose

$$(L_n - a\delta_{n,0})|phys\rangle = 0 \qquad (\tilde{L}_n - a\delta_{n,0})|phys\rangle = 0 \qquad n \geq 0$$

$\to$ $(L_0 - \tilde{L}_0)|phys\rangle = 0$ level matching condition

(due to the invariance $\sigma \to \sigma + a$)

$\sim$

Noether charge for $\delta\sigma^1 = \mathrm{const}$

($T_{1\mu} = (\partial_1 X^\mu)\,\dot{X}$ ...)

(This is similar to the Gupta-Bleuer quantization for the electromagnetic field. If we impose the Lorentz gauge $\partial \cdot A = 0$ the Maxwell eq becomes free eq for A^μ. But the usual quantization leads to negative norm states and we impose

$\partial_\mu A^{\mu -} |phys\rangle$ where $-$ denote the negative frequency part)

One have to figure out the constant a in $(L_0 - a)|phys\rangle = 0$

This involves $\frac{1}{2}\sum_m \alpha_{-m}\alpha_m = \frac{1}{2}\sum :\alpha_{-m}\alpha_m: + \frac{1}{2}\sum_{n=1}^{\infty} n$, for each direction of X^μ. The last term diverges and must be regularized. Here one uses the zeta function realization to obtain

$$\sum_{n=1}^{\infty} n \simeq \zeta(-1) = -1/12 \qquad \text{with } \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and using the analytic continuation of Riemann zeta function

For D dimension.

$$\frac{1}{2}\sum_m \alpha^\mu_{-m}\alpha_{m\mu} = \frac{1}{2}\sum :\alpha^\mu_{-m}\alpha_{m\mu}: + \frac{D-2}{2}\sum_{n=1}^{\infty} n$$

$$= \frac{1}{2}\sum :\alpha^\mu_{-m}\alpha_{m\mu}: - \frac{D-2}{24}$$

Here naively the zero point energy of X^0 cancels the contribution from one X^i with X^i being a space like direction. This counting will be justified using the light cone gauge to be explained shortly.

For $D=26$ we have

$$(L_0 - 1)|phys\rangle = (\tilde{L}_0 - 1)|phys\rangle = 0$$

$D=26$ is required for the decoupling of the negative norm states in the conformal gauge.

In doing that the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)$$ plays an important role in figuring this out.

quantum part
(normal ordering effect)

Here we just accept that and look for the spectrum in $D=26$.

Define $N \equiv L_0 - \frac{1}{2}(\alpha_0)^2 = \sum_{n>0} \alpha_{-n}\cdot\alpha_n = L_0 - \frac{\alpha'}{4} p^2$ ($p^2 = -m^2$, mass of the state)

$$N - a = \underbrace{L_0 - a}_{=0} + \frac{\alpha'}{4} m^2$$

$$m^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)$$

for $D=26$ $\quad m^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-1)$

$N = \tilde{N} = 0 \qquad |0\rangle \qquad m^2 = -4/\alpha' \quad$ tachyon

$N = \tilde{N} = 1 \qquad \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0\rangle \qquad m^2 = 0 \quad$ massless modes

space-time two rank tensors $\rightarrow G^{\mu\nu}, B^{\mu\nu}, \phi$

graviton!

Thus we learn that only in $D=26$,

we can have the massless modes in the string spectrum

one of which can be interpreted as a graviton.

It's interesting that $D=26$ is picked up as a quantum consistency condition i.e., decoupling of the negative norm state in the conformal gauge.

Now let's comment on the light cone gauge. The light cone gauge is the gauge where we can have just the physical degrees of freedom.

Using the residual gauge invariance after the conformal gauge fixing, $\quad \sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$

One can choose the light cone gauge $\quad X^+(\sigma,\tau) = x^+ + \alpha' p^+ \tau$

where $X^+ \equiv (X^0 + X^{D-1})/\sqrt{2} \quad X^- \equiv (X^0 - X^{D-1})/\sqrt{2}$.

The Virasoro constraint equations $(\dot{X} \pm X')^2 = 0$ become

$$(\dot{X}^- \pm X'^-) = (\dot{X}^i \pm X'^i)^2 / 2\alpha' p^+ \qquad (\, V\cdot W = V^i W^i - V^+ W^- - V^- W^+$$

This equation can be solved for X^- in terms of X^i so that in the light cone gauge both X^+ and X^- can be eliminated leaving only the transverse oscillators X^i.

For $X^- = x^- + \alpha' p^- \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n^- e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^- e^{-in(\tau+\sigma)} \right)$

One obtains

$$\alpha_n^- = \sqrt{\frac{\alpha'}{2}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha^i_{n-m} \alpha^i_m$$

$$\tilde{\alpha}_n^- = \sqrt{\frac{\alpha'}{2}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \tilde{\alpha}^i_{n-m} \tilde{\alpha}^i_m$$

In particular for $n=0$

$$\alpha' p^+ p^- = \sum_i \sum_{m=-\infty}^{\infty} \alpha^i_{-m} \tilde{\alpha}^i_m = \frac{\alpha'}{2}(p^i)^2 + 2\left(\sum_i \sum_{n>0} \alpha^i_{-n} \alpha^i_n - a\right)$$

after considering normal ordering

$$M^2 = 2p^-p^+ - (p^i)^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)$$

which is the same expression we obtained before

As an next example, let us consider the compartification on S^1

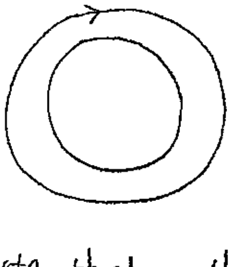
$$X \sim X + 2\pi R$$

Since the wave function should be periodic under $X \to X + 2\pi R$

$$e^{ip\cdot(2\pi R)} = 1 \qquad p = n/R$$

Furthermore string can wind around the circle

$X(2\pi R) = X(0) + 2\pi R m$ ← this is called the winding mode



$$X = x + \alpha' p \tau + mR\sigma + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n e^{-in(\tau-\sigma)} + \tilde{\alpha}_n e^{-in(\tau+\sigma)} \right)$$

Note that the spectrum is invariant under $R \to \alpha'/R$. One can see that the string interaction is also invariant under such transformation. This symmetry of the string theory is called the T-duality. One sees the first example of the "stringy" geometry. The geometry seen by a string could be quite different from the usual geometry seen by the field theory. In the string theory one cannot probe shorter than $\sqrt{\alpha'}$.

Partition function and the modular invariance for $\mathbb{R}^{25}\times S^1$

$$Z = \mathrm{Tr}\, q^{L_0-1}\, \bar{q}^{\tilde{L}_0-1} \qquad q = e^{2\pi i\tau}$$

$$= \mathrm{Tr}\, e^{-2\pi \,\mathrm{Im}\,\tau\, (\underbrace{L_0+\tilde{L}_0-2}_{\text{Hamiltonian}}) + 2\pi i\, \mathrm{Re}\,\tau\, (\underbrace{L_0-\tilde{L}_0}_{\text{level matching condition}})}$$

T_{++}
$= \frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu$
$= \sum_n L_n e^{-in(\tau-\sigma)}$
$+ \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}$

$$\to \quad L_0 = \frac{\alpha'}{4}\left(\frac{n}{R}+\frac{mR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\tilde{p}^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n$$

$\tilde{p}$: remaining 25 directions

$$\tilde{L}_0 = \frac{\alpha'}{4}\left(\frac{n}{R}-\frac{mR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\tilde{p}^2 + \sum_{n>0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

Note that $L_0+\tilde{L}_0-2 = \sum_i \frac{\alpha'}{2}(p^2+M_i^2)$ — p: all 26 directions

using $N-a = \frac{\alpha'}{4}m^2$

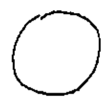
One loop vacuum amplitude

$$Z = \int \mathcal{D}\phi\, \exp -\phi(p^2+m^2)\phi = \frac{1}{\sqrt{\det p^2+m^2}} = e^{-\Lambda}$$

particle

$$\Lambda = \frac{1}{2}\ln\det(p^2+m^2) = \frac{1}{2}\int\frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2)$$

$$= \int_0^\infty \frac{dt}{2t}\int\frac{d^Dp}{(2\pi)^D}\, e^{-t(p^2+m^2)}$$

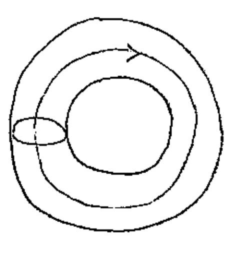


String

$$\int\frac{d\tau\, d\bar{\tau}}{2\,\mathrm{Im}\,\tau}\,\mathrm{Tr}\, e^{-2\pi\,\mathrm{Im}\,\tau\,(L_0+\tilde{L}_0-2) + 2\pi i\,\mathrm{Re}\,\tau\,(L_0-\tilde{L}_0)}$$

$$\| $$

$$\int\frac{d^Dp}{(2\pi)^D}\sum_{M_i} e^{-\alpha'\pi\,\mathrm{Im}\,\tau\,(p^2+M_i^2)}$$



Oscillator parts (concentrate one direction)

$$\mathrm{Tr}\, q^{L_0} \sim \sum e^{2\pi i\tau \sum_{n>0}\alpha_{-n}\cdot\alpha_n}$$

$$= \prod_{n=1}^{\infty}\sum e^{2\pi i\tau\, n a_n^\dagger a_n} \quad \leftarrow \text{occupation \#}$$

~ torus amplitude

$$\| \quad 1 + e^{2\pi i\tau} + e^{4\pi i\tau} + \cdots$$

$$= \prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i\tau n}} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

$$Z = \int\frac{d\tau\, d\bar{\tau}}{2\,\mathrm{Im}\,\tau}\,\mathrm{Tr}\, q^{L_0-1}\,\bar{q}^{\tilde{L}_0-1}$$

$$= \int\frac{d\tau\, d\bar{\tau}}{2\,\mathrm{Im}\,\tau}\int\frac{d^{25}p}{(2\pi)^{25}}\,\mathrm{Tr}\, e^{-2\pi\,\mathrm{Im}\,\tau\,(L_0+\tilde{L}_0-2) + 2\pi i\,\mathrm{Re}\,\tau\,(L_0-\tilde{L}_0)}$$

$$= \int\frac{d\tau\, d\bar{\tau}}{2\,\mathrm{Im}\,\tau}\,\frac{1}{(\alpha'\,\mathrm{Im}\,\tau)^{25/2}}\,\frac{\sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2}\,\bar{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2}}{|\eta(\tau)^{24}|^2}.$$

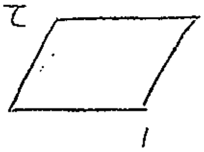
Here we define $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$

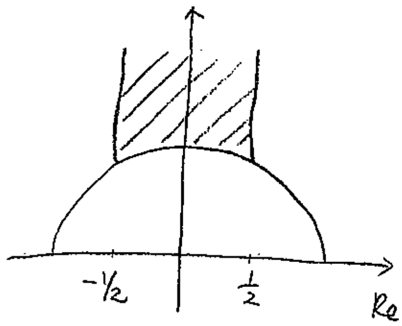
and we set $\alpha' = 2$ for the convenience

Note that the momentum integration is done over 26-dimension while the oscillator term integration is done over 24-dimension just the transverse directions. Recall that in the light cone gauge the zero mode momentums are still there in the light cone directions while the oscillators are expressed in terms of those in the transverse directions.

Modular invariance

torus $z \sim z+1 \sim z+\tau$





fundamental domain

one can characterize the shape of a torus by a complex parameter τ

The operations $T : \tau \to \tau + 1$

$S : \tau \to -1/\tau$

generate large diffeomorphisms of the torus $\sim SL(2.\mathbb{Z})$

(i.e diffeomorphisms not connected with the identity)

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2.\mathbb{Z})$$

Thus the partition function should be invariant under $SL(2.\mathbb{Z})$ or T/S

It's known that
$$\eta(\tau+1) = \exp \frac{\pi i}{12}\, \eta(\tau)$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\; \eta(\tau)$$

Under T $\quad \tau \to \tau + 1$

$$\frac{\sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2}}{\left| q^{1/24} \prod_n (1-q^n) \right|^2} \; e^{\pi i \frac{2n}{R} \cdot mR}$$

$$e^{\pi i \frac{2n}{R} \cdot mR} = e^{2\pi i mn} = 1$$

winding modes form the dual lattice to those of momentum modes

The last term indicates that their inner product is always an even integer

Under S, using the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}^p} \exp -\pi (n+x) \cdot A \cdot (n+x) = (\det A)^{-1/2} \sum_{m \in \mathbb{Z}^p} \exp(-\pi m A^{-1} m + 2\pi i m \cdot x)$$

positive definite

with $x=0$, $A = \begin{pmatrix} 2\frac{\mathrm{Im}\,\tau}{R^2} & i\,\mathrm{Re}\,\tau \\ i\,\mathrm{Re}\,\tau & 2(\frac{R}{2})^2 \end{pmatrix}$

and $A^{-1} = \begin{pmatrix} 2(\frac{R}{2})^2\, \mathrm{Im}(-1/\tau) & i\,\mathrm{Re}\,-1/\tau \\ i\,\mathrm{Re}\,-1/\tau & 2\frac{1}{R^2}\, \mathrm{Im}(-1/\tau) \end{pmatrix}$ $\left(\begin{array}{l} \text{as } \tau \to -1/ \\ R \to 2/ \end{array} \right.$

One can see that the expression is modular invariant

This is a special example of self-dual lattice

winding and momentum modes are interchanged

dual lattice

winding + momentum $\to$ self-dual lattice

One sees the simplest example of an <u>even self-dual lattice</u>, which is crucial for the modular invariance.

Using the above results, one can directly see that Z is indeed invariant under S and T.

proof of

Poisson resummation formula

$$f(x) = \sum_{n\in\mathbb{Z}^p} \exp\left[-\pi (n+x)\cdot A\cdot (n+x) \right]$$

positive definite $p\times p$ matrix

$f(x)$ is periodic in x with integer period

$$f(x) = \sum_{m\in\mathbb{Z}^p} e^{2\pi i m\cdot x}\, g(m)$$

$$g(m) = \int_0^1 d^px\; e^{-2i\pi \vec{m}\cdot\vec{x}}\, f(x)$$

$$g(m) = \sum_{n\in\mathbb{Z}^p} \int_0^1 d^px\; \exp\left(-\pi (n+x)\cdot A\cdot (n+x) - 2\pi i m x \right)$$

$$= \int_{-\infty}^{\infty} d^px\; \exp -\pi x\cdot A\cdot x - 2\pi i m\cdot x$$

$$= (\det A)^{-1/2} \exp -\pi\, m\cdot A^{-1}\cdot m$$

$$\therefore\; f(x) = (\det A)^{-1/2} \sum_{m\in\mathbb{Z}^p} \exp\left(-\pi m\cdot A^{-1}\cdot m + 2\pi i m\cdot x \right)$$

Open string theory

$$S = \frac{1}{2\pi\alpha'}\int_0^\pi \cancel{d\tau}\, d\sigma \int d\tau \sqrt{-\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$\underset{\gamma^{ab}=\eta^{ab}}{=} \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \int d\tau \left(-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu \right)$$

eq. of motion $(-\partial_\tau^2 + \partial_\sigma^2) X^\mu = 0$

boundary condition $\delta X^\mu \partial_\sigma X_\mu \big|_{0,\pi} = 0$

$X^\mu =$ constant Dirichlet

$\partial_\sigma X^\mu = 0$ Neumann

Choose the Neumann B.C.

$$X^\mu(\sigma,\tau) = x^\mu + \underset{\uparrow}{2\alpha' p^\mu \tau} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

identified with the total momentum $= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \frac{dX^\mu(\sigma)}{d\tau}$

$$L_m = \frac{1}{2\pi\alpha'}\int_0^\pi \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--} \right) d\sigma = \frac{1}{2}\sum \alpha_{m-n}\cdot\alpha_n$$

$$H = L_0 = \frac{1}{2}\sum \alpha_{-n}\cdot\alpha_n \quad \text{with} \quad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

conditions on the physical state

$$L_m |\text{phys}\rangle = 0 \qquad (L_0 - \underset{\uparrow}{a}) |\text{phys}\rangle = 0$$

$1/24$ per physical boson

$$L_0 = \alpha' p^2 + \underbrace{\sum_{n>0} \alpha_{-n}\cdot\alpha_n}_{\equiv N}$$

$$m^2 = \frac{1}{\alpha'}(N-a) = \frac{1}{\alpha'}(N-1) \quad \text{in} \quad D=26$$

Spectrum

$N=0$; $|0\rangle$ $m^2 = -1/\alpha'$ tachyon $V = \exp ik\cdot X$

$N=1$ $a_{-1}^\mu|0\rangle$ $m^2 = 0$ gauge boson # of physical state $D-2$

~ gauge symmetry

$$V = \underset{(SM)}{\partial_\tau X^\mu} \exp ik\cdot X$$

For imposing Dirichlet condition at both ends

with $X^\mu|_0 = x_1^\mu \qquad X^\mu|_\pi = x_2^\mu$

$$X^\mu(\sigma, \tau) = x_1^\mu + \frac{x_2^\mu - x_1^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin n\sigma$$

(no momentum p)

Contribution for H and L_0 from X^μ

$$H = L_0 \leftarrow \frac{1}{4\pi\alpha'}(X_2^\mu - X_1^\mu)^2 + \sum_{n>0}\alpha_{-n}^\mu\alpha_n^\mu$$

generally (i Dirichlet direction μ: Neumann direction)

$$H = L_0 = (\alpha' p^\mu)^2 + \frac{1}{4\pi\alpha'}(X_2^i - X_1^i)^2$$
$$+ \sum_{n>0}\alpha_{-n}^\mu\alpha_n^\mu + \sum_{n>0}\alpha_{-n}^i\alpha_n^i$$

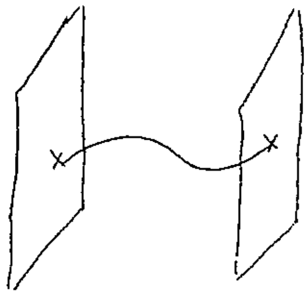
intersecting brane spectrum ?

the meaning of boundary conditions ?

vertex op $\partial_\sigma X^\mu \exp ik\cdot X$

→ clear in the context of D-branes

D-branes : hyperplane on which open string can end



$i \qquad j$

D3-brane world volume

x^0, x^1, x^2, x^3 ← impose Neumann

transverse location

e.g. $x^4 = x^5 = \cdots = 0$

$x^4 = L \quad x^5 = x^6 = \cdots = 0$ impo. Dirich

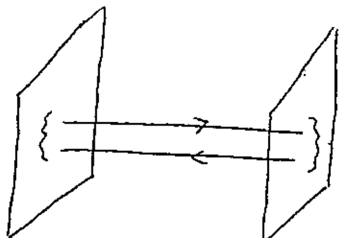
general open string states are described by

~~14. λij~~ $|4, ij\rangle \lambda_{ji}$ λ is hermitian (reality condit of the string ? field)

↑ world sheet fields ↑ Chan Paton factors (carry gauge group)

i, j : D-brane label

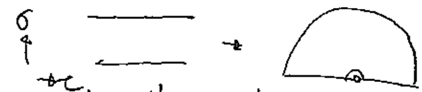
if coincident then it's easier

$U(1)$  $U(1)$ →  $U(2)$ gauge theory

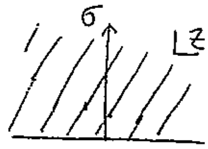
massive gauge bosons

Coincident n D-branes ~ U(n) gauge theory

What is the eqn of motion for an abelian gauge field coupled to the Bosonic open string? conformal tra



The action for the string coupled on the boundary to the electromagnetic field is (M^2 upper half plane)



$$S = \frac{1}{2\pi\alpha'}\left[\frac{1}{2}\int_{M^2} d^2z\, \partial^a X_\mu \partial_a X^\mu + i\int_{\partial M} d\tau\, A_\mu \partial_\tau X^\mu\right] \quad -(1)$$

$z = \tau + i\sigma$

where A_μ has been rescaled to contain a factor $2\pi\alpha'$.

In this calculation we will use the background field approach. This is the standard calculation in string theory. The basic idea is that the string theory has the conformal invariance. Thus if we compute the beta function β^A for the electromagnetic field A_μ, we should have $\beta^A = 0$

(different scale factor at different points

If we expand the action (2.1) around arbitrary background $\bar{X}$

$$X^\mu(\tau,\sigma) = \bar{X}^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma)$$

we get

$$A^\mu = A^\mu_0 + \partial_\nu A_\mu \xi^\nu + \frac{\partial_\alpha\partial_\beta A_\mu}{2!}\xi^\alpha\xi^\beta + \cdots$$

cf) $\frac{1}{g^2}\int F^2$ g does not run if the theory is scale invariant

$$S[\bar{X}+\xi] = S[\bar{X}]$$

$$+ \frac{1}{2\pi\alpha'}\Big[\int_{M^2} d^2z\, (\partial^a \bar{X}_\mu \partial_a \xi^\mu + \frac{1}{2}\partial^a\xi_\mu\partial_a\xi^\mu)$$

$$+ i\int_{\partial M} d\tau\, (F_{\mu\nu}\xi^\mu\partial_\tau \bar{X}^\nu + \frac{1}{2}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_\tau\bar{X}^\mu$$

$$+ \frac{1}{2}F_{\mu\nu}\xi^\nu\partial_\tau\xi^\mu + \frac{1}{3}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_\tau\xi^\mu + \cdots)\Big]$$

where $\nabla_\nu = \frac{\partial}{\partial \bar{X}^\nu}$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

assume that the fields vary slowly.

→ neglect terms with more than one deriv of F

On shell, the terms linear in ξ disappear since $\bar{X}$ satisfies the eq of motion

$$\Omega \bar{X}^\mu = 0 \qquad (\Omega = \partial_\tau^2 + \partial_\sigma^2)$$

mode expansion?

$$\partial_\sigma \bar{X}^\mu + i F_\nu{}^\mu \partial_\tau \bar{X}^\nu\big|_{\partial M} = 0$$

The on-shell action is

$$S[\hat{X}+\xi] = S[\hat{X}] + \frac{1}{2\pi\alpha'}\int_{M^2} d^2z \, \frac{1}{2}\partial^a\xi_\mu \partial_a \xi^\mu$$

$$+ \frac{\lambda}{2\pi\alpha'}\int_{\partial M} d\tau \Big(\frac{1}{2}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda \partial_\tau \hat{X}^\mu + \frac{1}{2}F_{\mu\nu}\xi^\nu\partial_\tau\xi^\mu$$

$$+ \frac{1}{3}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_\tau\xi^\mu + \cdots \Big)$$

Now compute the field-theory 1-loop counter term to the gauge coupling term in ①, namely

$$\Delta S_c(\hat{X}) = \frac{\lambda}{2\pi}\int_{\partial M} d\tau \, \Gamma_\mu \partial_\tau \hat{X}^\mu$$

~~From~~ The Neumann propagator in the upper half plane

$$\frac{1}{2\pi\alpha'}\nabla^2 G(z,z') = -\delta(z-z')$$

$$\partial_\sigma G(z,z')\Big|_{\sigma=0} = 0 \qquad \text{with } z = \tau + i\sigma$$

The solution can be easily obtained by the image method to

$$G(z,z') = -\alpha'\left(\ln|z-z'| + \ln|z-\bar{z}'|\right)$$

We can compute the counter term with this propagator and sum up all 1-loop graphs with an external $\partial_\tau\hat{X}$ and all possible insertions of the vertex $F_{\mu\nu}\xi^\nu\partial_\tau\xi^\mu$.

More straightforward method is to compute the exact propagator in the presence of the gauge field F.

The B.C. is changed,

$$\partial_\sigma G(z,z')_{\mu\nu} + \lambda F_\mu{}^\lambda \partial_\tau G_{\lambda\nu}(z,z')\Big|_{\sigma=0} = 0$$

The solution is

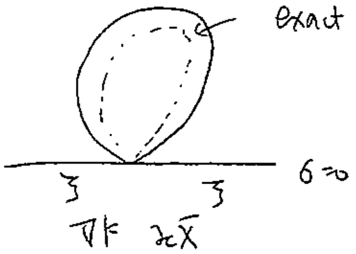
← meaning of this ?

$$G_{\mu\nu}(z,z') = -\alpha'\Big[\delta_{\mu\nu}\ln|z-z'| + \frac{1}{2}\left(\frac{1-F}{1+F}\right)_{\mu\nu}\ln(z-\bar{z}')$$

$$+ \frac{1}{2}\left(\frac{1+F}{1-F}\right)_{\mu\nu}\ln(\bar{z}-z')\Big]$$

(Symmetric under $z \leftrightarrow z'$, $\mu \leftrightarrow \nu$)

The only counterterm to S_Σ is



$$\Delta S_\Sigma = \frac{-i}{2\pi\alpha'} \int d\tau \, \frac{1}{2} \nabla_\nu F_{\mu\lambda} \partial_\tau \bar{X}^\mu G^{\nu\lambda}(\tau,\tau')$$

propagator on the boundary ($\sigma=\sigma'=0$)

In the limit $\tau \to \tau'$

$$G_{\nu\lambda}(\tau\to\tau') = -\alpha' \left[1 + \frac{1}{2}\frac{1-F}{1+F} + \frac{1}{2}\frac{1+F}{1-F} \right]_{\nu\lambda} \ln \Lambda$$

$$= -2\alpha' \ln \Lambda \, (1-F^2)^{-1}_{\nu\lambda}$$

Λ : short distance cutoff

$$\therefore \quad \beta_\mu{}^A = \Lambda \frac{\partial}{\partial \Lambda} \Gamma_\mu = \nabla^\nu F_\mu{}^\lambda (1-F^2)^{-1}_{\lambda\nu}$$

eqn of motion $\quad \nabla^\nu F_\mu{}^\lambda (1-F^2)^{-1}_{\lambda\nu} = 0 \quad$ — ②

positive definite

(F is actually $2\pi\alpha' F$ → ② contains all orders in α')

At the leading order in α' → Maxwell's equation

The beta function is not derivable from any action but $\chi_{\mu\nu} \beta^\nu_A$ is for a suitable $\chi_{\mu\nu}$

$$(1-F^2)^{-1}_{\mu\nu} \beta^\nu_A = \nabla^\nu \left(\frac{F}{1-F^2}\right)_{\mu\nu} - \left(\frac{F}{1-F^2}\right)_{\mu\lambda} \nabla^\nu F^{\lambda\rho} \left(\frac{F}{1-F^2}\right)_{\rho\nu}$$

Bianchi identity →
$$= \nabla^\nu \left(\frac{F}{1-F^2}\right)_{\mu\nu} + \frac{1}{4}\left(\frac{F}{1-F^2}\right)_{\mu\nu} \nabla^\nu \operatorname{tr} \ln (1-F^2)$$

This is derivable from

$$\mathcal{L}_{eff} = \exp \frac{1}{4} \operatorname{tr} \ln (1-F^2) = \exp \frac{1}{2} \operatorname{tr} \ln (1+F) = \sqrt{\det(1+F)}$$

whose Euler-Lagrange eqs are

$$\sqrt{\det(1+F)}\, (1-F^2)^{-1}_{\mu\nu} \beta^\nu_A = 0$$

In the leading order in α', $\mathcal{L}_{eff}$ is reduced to the action for the Maxwell theory.

Combined with

$$S_2 = \frac{1}{2\pi\alpha'}\int d^2\sigma\ \varepsilon^{\alpha\beta}\ \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)$$

with $\int_{\partial M} A = \int_{\partial M} \frac{dX^\mu}{d\tau} A_\mu$

we have $\frac{1}{2\pi\alpha'}\int B + \int_{\partial M} A$ (using the differential form notation)

This is invariant under the vector gauge transformation

$\delta A = d\Lambda$ ($\delta A^\mu = \partial^\mu \Lambda$) $\leftarrow$ ($\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$)

but the two-form gauge transf. $\delta B = d\xi$ gives a surface term cancelled by assigning a transformation $\delta A = -\xi/2\pi\alpha'$ ($\delta A^\mu = -\frac{\xi^\mu}{2\pi\alpha'}$)

Thus the combination $B + 2\pi\alpha' F$ is invariant under both transformation

This implies that the Born-Infeld action we obtain is modified

$$S = \int \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}$$

$$G_{ab} = \partial_a X^\mu \partial_\beta X^\nu G_{\mu\nu} \quad \text{(pull back)}$$

$$B_{ab} = \partial_a X^\mu \partial_\beta X^\nu B_{\mu\nu}$$

(cf) Nambu-Goto action $\int\sqrt{-\det G_{ab}}$ is classically equivalent to string action we use $\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu$

The dilaton coupling is simply

$$S = \int d^{p+1}x\ e^{-\phi}\sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}$$

Since this is an open string tree level action

Previously we saw that closed string perturbation has all the Riemann surfaces.

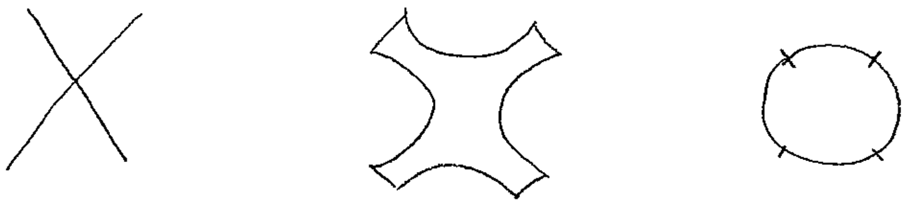
In the case of oriented open string, we should consider Riemann surfaces with boundary. The expansion parameter is

$$e^{\phi(-2+2h+b)}$$

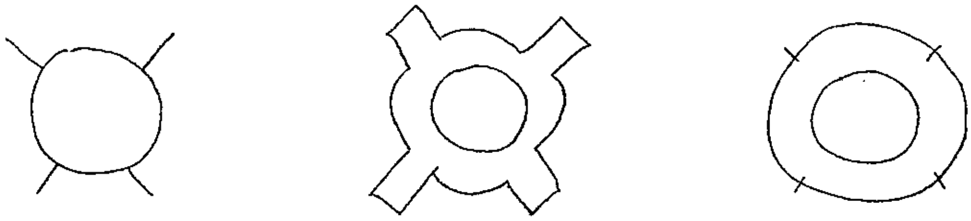
b: # of boundary

h: # of handles

The lowest diagram contains $e^{-\phi}$ $\rightarrow$ open string tree level.



1 baundary



2 baundaries

Previously, in the presence of the background electromagnetic field, we derive the eqn of motion for the massless modes of the open string.

We can also derive the eqn of motion for the closed string massless modes

If we consider the general gravitational background $g_{\mu\nu}$

$$S = \frac{1}{\cancel{4}\,2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \underset{\substack{\downarrow \\ g_{\mu\nu}}}{\eta_{\mu\nu}}$$

Write down the most general action for $X^\mu(\sigma,\tau)$ that is invariant under reparametrization of the string world sheet and renormalizable by power counting

$\Downarrow$

two derivative ~~in each ter~~

$$S_1 = \frac{1}{\cancel{4}\,2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X^\rho)$$

$$S_2 = \frac{1}{\cancel{4}\,2\pi\alpha'} \int d^2\sigma\, \epsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho)$$

$\leftarrow$ string couples to 2- particle " 1-

pull back of the spacetime $g_{\mu\nu}$, $B_{\mu\nu}$

~~The~~ One interesting effect of S_2 ~~is~~

if we consider $\frac{1}{2\pi\alpha'} \int B + \int_{\partial M} A$

using the differential form notation

This is invariant under the vector gauge transf.

$\delta A = d\Lambda$

but the two-form gauge transf. $\delta B = d\xi$ gives a surface term cancelled by assigning a transformation $\delta A = -\xi/2\pi\alpha'$

Thus the combination $B + 2\pi\alpha' F$ is invariant under both transformations.

This implies that the Born-Infeld action we obtain is modified

$$S = \int \sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)}$$

$$G_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad \text{(pull back)}$$

and similar for other terms

Now what is the dilaton coupling?

It is simply $S = \int d^{p+1}x \, e^{-\phi} \sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)}$

since this is an open string tree level action

To understand this, we need some understanding of the string perturbation expansion

The dilaton appears in the world sheet as

$$S_3 = \frac{1}{4\pi} \int d^2\sigma \sqrt{h}\, \Phi(X) R^{(2)}$$

Note that $\frac{1}{4\pi}\int d^2\sigma \sqrt{h} R^{(2)}$ is a top'l invariant

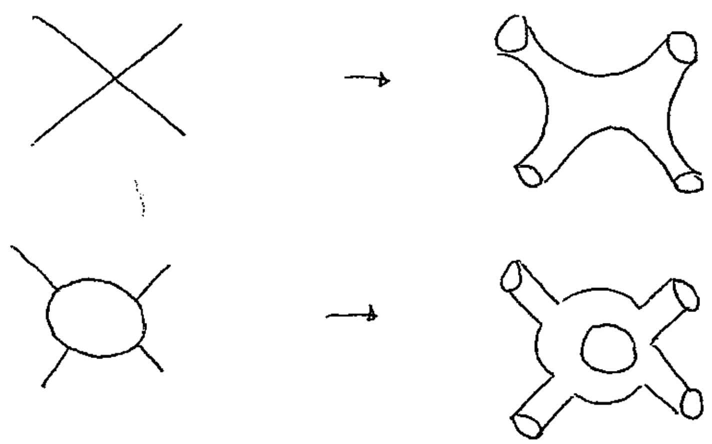
$$= \chi = 2(1-g)$$

g genus of the Riemann surface

In the effective action, this induces a term

$$e^{-S_3} \sim e^{-\phi 2(1-g)} = e^{\phi(-2+2g)} \quad - (*)$$

Now e^{ϕ} is the coupling constant for the string perturbation theory



$+ \cdots \qquad + \cdots$

One can see that string diagrams involve all the Riemann surfaces and if we have many handles, the corresponding diagrams describe higher loops.

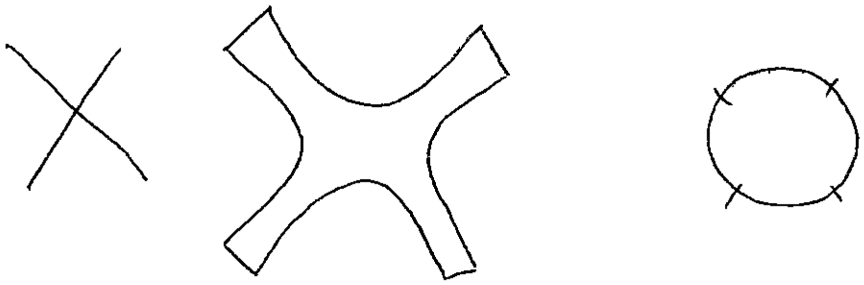
This is indeed captured by ①.

If we consider oriented open string, we should consider Riemann surfaces with boundary.

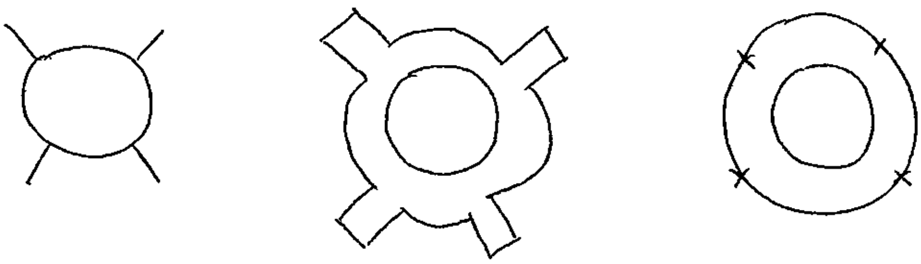
In this case, the expansion parameter $\sim e^{\phi(-2+2h+b)}$ (# of bdy: b; # of handles: h)

the lowest diagram contains e^{ϕ}

→ open string tree level



1 boundary



2 boundaries

One can similarly compute the β function for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, to find the eqn. of motion

The result is

$$R_{\mu\nu} + \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi = 0 \quad ?$$

$$D_\lambda H^\lambda{}_{\mu\nu} - 2 (D_\lambda \Phi) H^\lambda{}_{\mu\nu} = 0$$

$$4 (D_\mu \Phi)^2 - 4 D_\mu D^\mu \Phi + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$$

D_μ : covariant derivative in D-dimension

$$\int \sqrt{-g}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) d^D x \quad -①$$

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \qquad \tilde{g}^{ab} = \Omega^{-2} g^{ab}$$

$$\tilde{R} = \Omega^{-2}\left(R - 2(D-1) g^{ac} \nabla_a \nabla_c \ln\Omega - (D-2)(D-1) g^{ac} \nabla_a \ln\Omega \nabla_c \ln\Omega\right)$$

$$① = \int \Omega^{-D} \sqrt{-\tilde{g}}\, e^{-2\Phi} \Big(\Omega^2 \tilde{R} + 2(D-1)\Omega^2 \tilde{g}^{ac} \nabla_a \nabla_c \ln\Omega$$

$$+ (D-2)(D-1)\Omega^2 \tilde{g}^{ac} \nabla_a \ln\Omega \nabla_c \ln\Omega + 4\Omega^2 \tilde{g}^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$$

$$- \frac{1}{12} \Omega^6 \tilde{g}^{\mu\mu'} \tilde{g}^{\nu\nu'} \tilde{g}^{\lambda\lambda'} H_{\mu\nu\lambda} H_{\mu'\nu'\lambda'} \Big)$$

$$= \int \sqrt{-\tilde{g}} \Big(\Omega^{-D+2} e^{-2\Phi} \tilde{R}$$

$$+ \Omega^{-D+2} e^{-2\Phi} \big(2(D-1) \tilde{g}^{ac} \nabla_a \nabla_c \ln\Omega + (D-2)(D-1) \tilde{g}^{ac} \nabla_a \ln\Omega \nabla_c \ln\Omega$$

$$+ 4 \tilde{g}^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$$

$$+ \Omega^{-D} e^{-2\Phi} \left(-\frac{1}{12}\right) \Omega^6 H^2 \Big)$$

coefficient of $\tilde{R}$ $= 1$ $\rightarrow$ $\Omega^{D-2} = e^{-2\Phi}$ $\qquad \Omega = e^{\frac{-2}{D-2}\Phi}$

$$= \int \sqrt{-\tilde{g}} \left(\tilde{R} - 4 \frac{1}{D-2} (\nabla\Phi)^2 - \frac{1}{12} e^{-\frac{8}{D-2}\Phi} H^2 \right) d^D x$$

$$= \int \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2} (\nabla\Phi)^2 - \frac{1}{12} e^{-\Phi} H^2 \right) d^{10} x \quad \text{for } 10\text{-}D.$$

(d) $\tilde{g}_{ab} = \Omega^2 g_{ab}$

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d + 2 \delta^d{}_{[a} \nabla_{b]} \nabla_c \ln\Omega - 2 g^{de} g_{c[a} \nabla_{b]} \nabla_e \ln\Omega$$

$$+ 2 \nabla_{[a} \ln\Omega\, \delta^d{}_{b]} \nabla_c \ln\Omega - 2 \nabla_{[a} \ln\Omega)\, g_{b]c}\, g^{df} \nabla_f \ln\Omega$$

$$- 2 g_{c[a} \delta^d{}_{b]} g^{ef} (\nabla_e \ln\Omega) \nabla_f \ln\Omega$$

$$\tilde{R}_{ac} = R_{ac} - (D-2) \nabla_a \nabla_c \ln\Omega - g_{ac} g^{de} \nabla_d \nabla_e \ln\Omega$$

$$+ (D-2)(\nabla_a \ln\Omega) \nabla_c \ln\Omega - (D-2) g_{ac} g^{de} (\nabla_d \ln\Omega) \nabla_e \ln\Omega$$

contract with $\tilde{g}^{ac}$ $\quad \tilde{R} = \Omega^{-2}\left(R - 2(D-1) g^{ac} \nabla_a \nabla_c \ln\Omega - (D-2)(D-1) g^{ac} \nabla_a \ln\Omega \nabla_c \ln\Omega \right.$

This can be derived from the 26-D action

$$S_{26} = -\frac{1}{2k^2}\int d^{26}x\,\sqrt{-g}\;e^{-2\Phi}\left(R + 4 D_\mu\Phi D^\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)$$

$- \cdots$

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial\Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta}$$

$G_{\mu\nu} = \cdots T_{\mu\nu}$

* T-duality and the D-branes

We have seen that if the string is compactified on a circle

$$X = x + \alpha'\frac{n}{R}\tau + mR\sigma + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{i}{n}\left(\alpha_n e^{-in(\tau-\sigma)} + \tilde{\alpha}_n e^{-in(\tau+\sigma)}\right) \quad ②$$

$$L_0 = \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n$$

$$\tilde{L}_0 = \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \sum_{n>0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

$$H = L_0 + \tilde{L}_0 = \frac{\alpha'}{2}\left(\left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2\right) + \sum_{n>0}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right)$$

The spectrum is invariant under $R \leftrightarrow \alpha'/R$

This means that just by looking at the spectrum, we cannot distinguish the string theory compactified on R and that compactified on α'/R.

② can be written in the more suggestive form

$$X = X_L(\tau+\sigma) + X_R(\tau-\sigma)$$

with

$$X_L(\tau+\sigma) = X_{L0} + \frac{1}{2}\overbrace{\left(\alpha'\frac{n}{R} + mR\right)}^{=\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0}(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{i}{n}\tilde{\alpha}_n e^{-in(\tau+\sigma)}$$

$$X_R(\tau-\sigma) = X_{R0} + \frac{1}{2}\underbrace{\left(\alpha'\frac{n}{R} - mR\right)}_{=\sqrt{\frac{\alpha'}{2}}\alpha_0}(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{i}{n}\alpha_n e^{-in(\tau-\sigma)}$$

$$\tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)$$

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)$$

The mass spectra of the theory at radius R and α'/R are identical when the winding and K-K modes are interchanged $n \leftrightarrow m$

$$\text{or} \quad \tilde{\alpha}_0 \to \tilde{\alpha}_0 \qquad \alpha_0 \to -\alpha_0$$

Thus if we write the radius R theory in terms of

$$X' = X_L(\tau+\sigma) - X_R(\tau-\sigma)$$

X' will describe the string theory on the radius α'/R.

The string spectrum is invariant under the space time parity transformation acting only on the right moving d.o.f.

Even though we will not go into detail, this transf. is a symmetry of the full string theory. This symmetry is called the T-duality.

It is important to note that T-duality acts non trivially on the dilaton. By the usual dimensional reduction, the effective 25-D coupling constant is $e^{\phi} R^{-1/2}$.

Duality requires this to be equal to $e^{\phi'} R'^{-1/2}$.

Thus $e^{\phi'} = e^{\phi} R^{-1} \alpha'^{1/2}$

The T-duality is a quite striking property of the string theory in contrast to the field theory.

If we start with D-dim field theory on the radius R and if we take $R \to 0$, we have $(D-1)$-dim field theory.

However if we shrink the radius R, the string theory still contains the information on the higher dimension compactified

On many instances of the string theory, the geometry seen by the string theory are quite different from the usual geometry seen by the field theory. T-duality is one aspect of such stringy geometry.

* T-duality of the oriented open string

Now consider the open string theory compactified on a circle.

Impose the Neumann B.C. on the circle.

The spectrum we have

$$X(\sigma,\tau) = x + 2\alpha' \frac{n}{R}\tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$$

Now the open string theory does not have the winding modes. It looks like the behavior is similar to the field theory in the $R\to 0$ limit. Apparently the compactified dimension disappears, leaving a theory in D-1 space dimension.

But the open string has always closed string as well (unitarity).

Let's carefully see what happens.

One thing we can see is that the boundary condition changes

$$\partial_\sigma X = 0 \quad \to \quad \partial_\tau X' = 0 \qquad \text{with} \quad X' = X_L - X_R$$

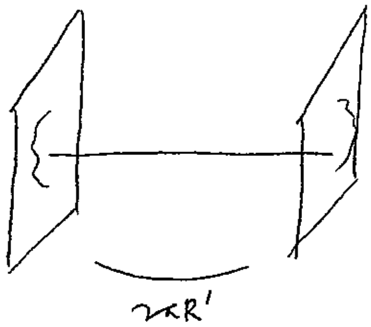
Thus we have the Dirichlet boundary condition.

The open string end points are living in 25-dim.

In fact, all end points are constrained to live on the same hyperplane

$$X'(\pi) - X'(0) = \int_0^\pi d\sigma\, \partial_\sigma X' = \int_0^\pi d\sigma\, \partial_\tau X$$

$$= 2\pi\alpha' p = 2\pi\alpha' \frac{n}{R} = 2\pi n R'$$



$X'(\pi)$ and $X'(0)$ are identified

Now study the effect of Chan-Paton factor.
Consider $U(N)$ oriented open string.

In compactifying X^{25} direction, we can include a Wilson line
$A_{25} = \mathrm{diag}(\theta_1, \theta_2 \ \ \theta_N)/2\pi R$, generically breaks $U(N) \to U(1)^N$
Locally this is pure gauge
$A_{25} = -i\Lambda^{-1}\partial_{25}\Lambda \qquad \Lambda = \mathrm{diag}(e^{iX\theta_1/2\pi R}, \ldots e^{iX\theta_N/2\pi R})$
One can gauge A_{25} away, but the gauge transf. is not periodic and the fields now pick up a phase

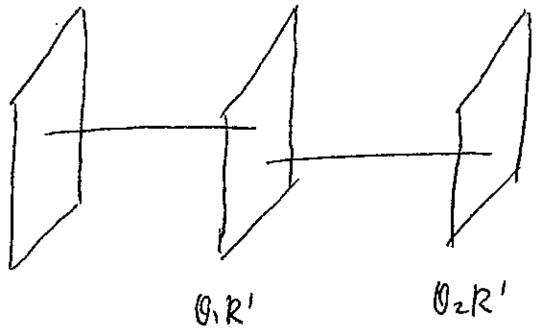
EM problem $\frac{(p-eA)^2}{2m}$ $e^{ip\cdot(2\pi R)}$ + ~~$2\pi R\theta$ term~~ $(\vec{p}-\frac{e\vec{A}}{c})^2$

$\mathrm{diag}(e^{-i\theta_1} \qquad e^{-i\theta_N}) \qquad ③$
under $X^{25} \to X^{25} + 2\pi R$

Due to the phase ③, the open string momentum now in general have fractional parts. In the dual picture, it implies the fractional windings, which means that endpoints are no longer on the same hyperplane.

Indeed, a string whose end points are in the state $|ij\rangle$ picks up a phase $e^{i(\theta_j - \theta_i)}$.
The momentum is $\frac{2\pi n + \theta_j - \theta_i}{2\pi R}$

Then $X'(\pi) - X'(0) = (2\pi n + \theta_j - \theta_i)R'$
The end point in state i is at
$X' = \theta_i R' = 2\pi\alpha' A_{25,ii}$



The mass spectrum

$$m^2 = p^2 + \frac{1}{\alpha'}(N-1) = \left(\frac{2\pi n + (\theta_1-\theta_2)R'}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1)$$

$2\pi n + (\theta_1-\theta_2)R'$ is the minimum length of a string winding between hyperplanes 1 and 2.

We have massless states for ~~the~~ 1=2

$\alpha^{\mu}_{-1}|k, 11\rangle \qquad V = \partial_t X^{\mu}$

$\alpha^{m}_{-1}|k, 11\rangle \qquad V = \partial_t X = \partial_n X'$

The first is a gauge field living on the hyperplane with (p+1) components to the hyperplane. The second is the gauge field in the compact direction. In the dual theory, it becomes the transverse position of the hyperplane.

More general gauge backgrounds correspond to curved surfaces. And the quanta of the gauge field corresponds to the fluctuations.

cf) the string in a flat background → massless closed string modes → fluctuation of the geometry

Here certain open string states correspond to the fluctuations of the shape

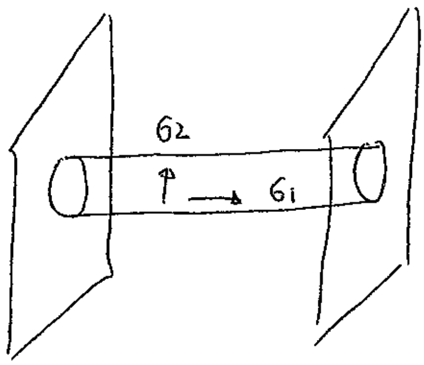
→ D-brane is a dynamical object

T-duality in a direction tangent to a Dp-brane
reduces it to a $(p-1)$-brane
T-duality " orthogonal "
reduces it to a $(p+1)$-brane

1

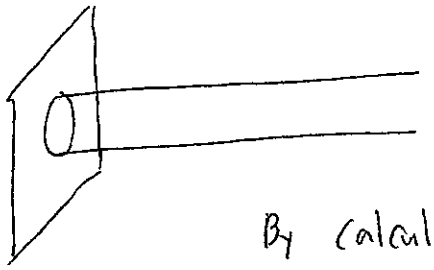
Open / closed string channel duality and the computation of the D-brane tension

It is instructive to compute the D-brane tension T_p.
(It is ~~pt~~ proportional to g^{-1} as noted above)
One could calculate it from the gravitational coupling to the D-brane, given by the disk with a graviton vertex operator
It's much easier to use the following trick.
Along with that, we introduce the open / closed string channel duality.
Consider two parallel Dp-branes at positions at $X'^{\mu}=0$ and $X'^{\mu}=Y^{\mu}$:



These two objects can feel each other's presence by exchanging closed strings.
This string graph is an annulus with no vertex operators

The poles from graviton and dilaton exchange give the coupling T_p of closed string state of the D-brane.
By conformal invariance, this can be seen either as an open string 1-loop or the tree of the closed string.
If the distance between two branes is large, the closed string exchange is dominated by the massless modes. (In computation, you will see that the IR limit of the closed string corresponds to the UV limit of the open string)

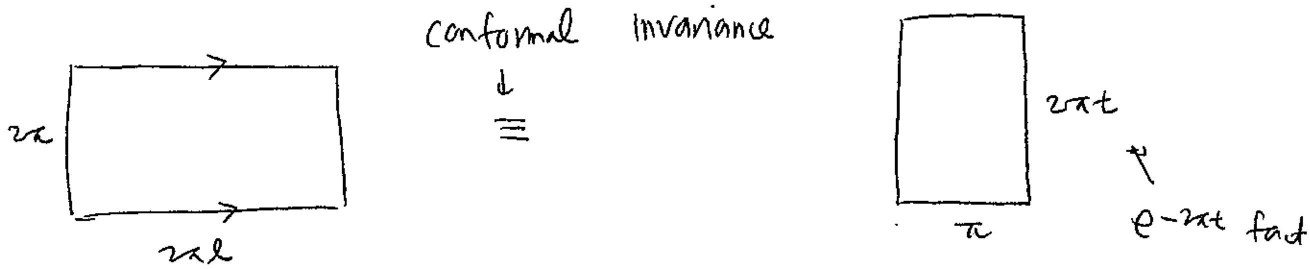


massless modes

$\sim \frac{1}{p^2}$ in 4-d

By calculating the cylinder diagram and using the open-closed string channel dualities, one can calculate the exchange of the massless modes

The relation between the parameters in the open string and the closed string channel



$$2\pi \times \frac{1}{2l} = 2\pi t \qquad t = \frac{1}{2l}$$

Calculation of the 1-loop vacuum amplitude or the 1-loop contribution to the cosmological constant in the effective action in the field theory for a bosoni particle with mass m

$$\Lambda = \frac{1}{2} \ln \det (p^2+m^2) = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln (p^2+m^2)$$

$$= \int_0^\infty \frac{dt}{2t} \int \frac{d^D p}{(2\pi)^D} e^{-t(p^2+m^2)}$$

Corresponding expression for the open string cylinder amplitude

$$\int_0^\infty \frac{dt}{2t} \mathrm{Tr}\, e^{-2\pi t (L_0 - 1)} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}\, e^{-2\pi t \alpha' (p^2 + m_\lambda^2)}$$

with $\alpha' m_\lambda^2 = N - 1 = \sum_{n>0} n a_n^+ a_n - 1 + \frac{Y \cdot Y}{4\pi^2 \alpha'}$

Thus we have

$$2V_{p+1}\int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}\, e^{-2\pi t(\alpha' p^2 + \sum_{n>0}\alpha_{-n}\alpha_n - 1 + \frac{Y\cdot Y}{4\pi^2\alpha'})}$$

$$= 2V_{p+1}\int_0^\infty \frac{dt}{2t}\,(8\pi^2 t\alpha')^{-\frac{p+1}{2}}\, e^{-\frac{Y\cdot Y t}{2\pi\alpha'}}\frac{e^{2\pi t}}{\prod_{n=1}^\infty (1-e^{-2\pi n t})^{24}}$$

using $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$

for $\eta(\tau) = q^{1/24}\prod_{n=1}^\infty (1-q^n) = e^{\frac{\pi i\tau}{12}}\prod_{n=1}^\infty (1-e^{2\pi i n\tau})$

$$= 2V_{p+1}\int_0^\infty \cancel{\frac{dt}{2t}}\,\frac{dl}{2l}\,(8\pi^2\alpha')^{-\frac{p+1}{2}}(2l)^{\frac{p+1}{2}}\, e^{-\frac{Y\cdot Y}{4\pi\alpha' l}}$$

$$\otimes\ \frac{e^{4\pi l}}{\prod_{n=1}^\infty (1-e^{-4\pi n l})^{24}}$$

$$\shortparallel\ (\underset{\text{tachyon}}{e^{4\pi l}} + \underset{\text{massless}}{24} + \cdots)$$

The massless pole is

$$A \sim V_{p+1}\frac{24}{2^{11}}(4\pi^2\alpha')^{11-p}\,\pi^{\frac{p-23}{2}}\,|Y|^{p-23}$$

$$= V_{p+1}\frac{24\pi}{2^9}(4\pi^2\alpha')^{11-p}\,G_{25-p}(Y^2)$$

$G_d(Y^2)$ massless scalar Green's function in d-dim.

This can be compared with a field theory calculation, the exchange of graviton plus dilaton between a pair of D-brane.

The propagator is from the bulk action

$$S = \int d^{26}x\left(\frac{1}{2\kappa^2}e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right)$$

and the couplings are from the D-brane action

$$S_p = -T_p\int d^{p+1}\xi\ e^{-\phi}\sqrt{\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab})}$$

One should be careful about the graviton and the dilaton mix, but in the end

$$A \sim \frac{D-2}{4}V_{p+1}T_p^2\kappa^2\,G_{25-p}(Y^2) \qquad T_p = \frac{\sqrt{2\pi}}{16\kappa}(4\pi^2\alpha')^{\frac{11-p}{2}}$$

We have the massless tadpole contribution

$$2V_{p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-\frac{p+1}{2}}\, e^{-\frac{Y\cdot Y t}{2\pi\alpha'}}\ 24$$

Integrate t $= V_{p+1}\frac{24}{2^{11}}(4\pi^2\alpha')^{11-p}\,\pi^{\frac{p-23}{2}}\,|Y|^{p-23}$

$= V_{p+1}\frac{24\pi}{2^{9}}(4\pi^2\alpha')^{11-p}\,G_{25-p}(Y^2)$

Where $G_{25-p}(Y^2)=\frac{1}{4}\pi^{\frac{p-25}{2}}\,\Gamma\left(\frac{23-p}{2}\right)(Y^2)^{\frac{p-23}{2}}$

is the scalar Green function in 9-p dimension

$(\Gamma(\frac{1}{2})=\sqrt{\pi})$

One can check that $G_3(Y^2)=\frac{1}{4\pi Y}$

(Calculation of the force

the force between the scalar charge

$S=\int -\frac{1}{2}(\partial\phi)^2+J\phi$

$-\partial_\alpha\partial^\alpha\phi=J \qquad \phi=\frac{J}{p^2} \qquad \rightarrow \qquad \frac{J_1J_2}{p^2} \qquad -\frac{\Sigma_1\Sigma_2}{r}$

$S=\int -\frac{1}{4}F^2+J_\mu A^\mu$

$-\partial_\mu(\partial^\mu A^\nu)=J^\nu \qquad A^\nu=\frac{J^\nu}{p^2} \qquad \rightarrow \qquad \frac{J_1{}^\nu J_{2\nu}}{p^2} \rightarrow \frac{-J_1{}^0J_2{}^0}{p^2} \qquad +\frac{Q_1Q_2}{r}$

gravity $\qquad -\Box h^{\mu\nu}=16\pi G T^{\mu\nu} \qquad h^{\mu\nu}\propto\frac{T^{\mu\nu}}{p^2} \qquad \frac{G\,T^{\mu\nu}T_{\mu\nu}}{p^2}=\frac{T_{00}T_{00}}{p^2} \qquad -\frac{GM_1M_2}{r}$

$\int -\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu}+h_{\mu\nu}T^{\mu\nu}$

Boundary State

Previously we saw that the open string 1-loop is factorized in the closed string channel and two D-branes exhibit closed string exchange.

This suggests that in the closed string channel the partition function of the open string can be written as the overlap of some states with the insertion of the closed string propagator

i.e. $\langle B_1 | e^{-\cancel{\frac{\pi}{t}}(L_0+\tilde{L}_0)t} | B_2 \rangle$

The states $|B_1\rangle$ and $|B_2\rangle$ are called the boundary states.

This is the very convenient tool to extract the D-brane coupling to the closed string state.

Now in the open string channel,

we have the boundary condition $\partial_\sigma X^\mu = 0$ ~~for~~ at $\sigma = 0$ N

$\partial_\tau X^\mu = 0$ D

When we go from open string to the closed string channel we interchange the role of σ and τ.

Thus $\partial_\tau X^\mu \big|_{\tau=0} = 0$ for N

$\partial_\sigma X^\mu \big|_\tau = 0$ for D

boundary states for toroidal compactification with B field ?

from the closed string mode expansion

$$X^\mu = x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \right)$$

with $\alpha_n^\mu \equiv i\sqrt{n}\, a_n^\mu$ $\quad \alpha_{-n}^\mu = -i\sqrt{n}\, a_n^{\mu\dagger}$

$$= x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{\sqrt{n}} \left(a_n^\mu e^{-in(\tau-\sigma)} + \tilde{a}_n^\mu e^{-in(\tau+\sigma)} + a_n^{\mu\dagger} e^{in(\tau-\sigma)} + \tilde{a}_n^{\mu\dagger} e^{in(\tau+\sigma)} \right)$$

$\partial_\tau X^\mu|_{\tau} = 0$ (N)

$\to$ $p^\mu = 0$ $\quad a^\mu_n e^{-i n \tau} = \tilde{a}^{\mu +}_n e^{i n \tau}$ $\quad a^\mu_n = \tilde{a}^{\mu +}_n e^{2 i n \tau}$

$\tilde{a}^\mu_n e^{-i n \tau} = a^{\mu +}_n e^{i n \tau}$ $\quad \tilde{a}^\mu_n = a^{\mu +}_n e^{2 i n \tau}$

This can be satisfied by the coherent state by noting that $a^\mu_n \sim \frac{\partial}{\partial a^{\mu +}_n}$

$$|B\rangle = \exp \sum_{n=1}^{\infty} (a^{\mu +}_n \tilde{a}^{\mu +}_n e^{2 i n \tau}) |0\rangle$$

$$= \exp \sum_{n=1}^{\infty} (-\tfrac{1}{n} \alpha^\mu_{-n} \tilde{\alpha}^\mu_{-n} e^{2 i n \tau}) |0\rangle$$

We can work out the boundary states corresponding to Dirichlet BC as well.

Now $|B, \tau\rangle = e^{i\tau(L_0 + \tilde{L}_0)} \exp \sum_{n=1}^{\infty} -\frac{1}{n} \alpha^\mu_{-n} \tilde{\alpha}^\mu_{-n} |0\rangle$

$$= e^{i\tau \underbrace{(L_0 + \tilde{L}_0)}_{= H}} |B, \tau = 0\rangle$$

This is just the time translation of the state at $\tau = 0$ and $e^{i\tau(L_0 + \tilde{L}_0)}$ is the propagator of the closed string.

The left eigenstate at the closed string state is given by the adjoint

$\langle 0| \exp - \sum_{n=1}^{\infty} \frac{1}{n} \alpha^\mu_n \tilde{\alpha}^\mu_n$ $\qquad$ ghost sectors

Now one can see that for a D-26 brane

open string partition function $\qquad$ exercise ?

$= \int_0^\infty dl \; \langle B| e^{-2l(L_0 + \tilde{L}_0 - 2)} |B\rangle$ $\qquad$ with $t = \frac{1}{2l}$

In many cases, the boundary state is used instead of D-branes especially before the concept of the D-brane is available.

Boundary states can be described in the abstract CFT, and this will describe the D-brane states in the abstract CFT

As another example we can work out the boundary state in the presence of the constant electromagnetic field

Open string boundary condition $\partial_\sigma X^\mu + F^{\mu\nu} \partial_\tau X_\nu = 0$

In the closed string channel, we impose $\partial_\tau X^\mu - F^{\mu\nu} \partial_\sigma X_\nu = 0$ at $\tau = \tau_0$
(for convenience we choose $\tau = 0$) $\quad \begin{pmatrix} \sigma' \\ \tau' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix}$

$$\to \quad (\eta^{\mu\nu} + F^{\mu\nu}) a_{\nu n} = (\eta^{\mu\nu} - F^{\mu\nu}) \tilde{a}^\dagger_{\nu n}$$

$$p^\mu = 0$$

$$|B\rangle = \exp \sum_{n=1}^{\infty} a^\dagger_{\mu n} \left(\frac{\eta - F}{\eta + F}\right)^{\mu\nu} \tilde{a}^\dagger_{\nu n} |0\rangle$$

$$= \underbrace{\ }_{T_p V_{p+1} \sqrt{-\det(\eta + F)}} \exp - \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{\mu -n} \left(\frac{\eta - F}{\eta + F}\right)^{\mu\nu} \tilde{\alpha}_{\nu, -n} |0\rangle$$

The normalization constant is chosen for the late convenience. This can also be calculated from the open string partition function and then match it with the boundary state result

One can compute the coupling of the D-brane with the massless fields of the closed string.

The graviton and the $B_{\mu\nu}$ are of the form

$$h_{\mu\nu} \tilde{\alpha}^\mu_{-1} \alpha^\nu_{-1} |p\rangle \quad \text{with} \quad h_{\mu\nu} = h_{\nu\mu} \quad p^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$$

$$\text{and} \quad B_{\mu\nu} \tilde{\alpha}^\mu_{-1} \alpha^\nu_{-1} |p\rangle \quad B_{\mu\nu} = -B_{\nu\mu} \quad p^\mu B_{\mu\nu} = 0$$

we extract the coupling from

$$\langle p | \tilde{\alpha}^\mu_{1} \alpha^\nu_{1} | B \rangle$$

For the graviton we obtain

phys sector

$$2T_p V_{p+1}\sqrt{-\det(\eta+F)}\,\left((\eta+F)^{-1}\right)^{\alpha\beta} h_{\beta\alpha}$$

and for the $B_{\mu\nu}$

compactified Boundary state

$$2T_p V_{p+1}\sqrt{-\det(\eta+F)}\,(\eta+F)^{-1\,\alpha\beta} B_{\beta\alpha}$$

(from $\left(\frac{\eta+F}{\eta+F}\right)^{\alpha\beta} h_{\beta\alpha} = \left(\frac{\eta+F}{\eta+F}\right)^{\alpha\beta} B_{\beta\alpha} = 0$

we have $\left(\frac{F}{\eta+F}\right)^{\alpha\beta} \begin{matrix} h_{\beta\alpha} \\ B_{\beta\alpha}\end{matrix} = -\left(\frac{1}{\eta+F}\right)^{\alpha\beta} \begin{matrix} h_{\beta\alpha} \\ B_{\beta\alpha}\end{matrix}$)

One can again see that the couplings ~~are~~ can be deduced from the Born Infeld action

$$S = -T_p\int d^{p+1}\xi\sqrt{-\det(G_{ab}+B_{ab}+(2\pi\alpha')F_{ab})}$$

by using $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $B_{\mu\nu}$

Similarly one can work out the coupling with the dilaton.

× Rotated and boosted brane

Let's consider T-dual configuration of the open string with the constant electromagnetic field

Starting from $\partial_\sigma X^\mu + F^{\mu\nu}\partial_\tau X_\nu = 0$

first consider the case with $F_{01} = f \neq 0$ only.

then the non trivial B.C.

$$\partial_\sigma X^1 - F_{01}\partial_\tau X^0 = 0$$

$$\partial_\sigma X^0 - F_{01}\partial_\tau X^1 = 0$$

Compactify on ~~T~~ X^1 direction and T-dualize.

The boundary condition is changed into

$$\partial_\tau(X^1 - F_{01}X^0) = 0$$

$$\partial_\sigma(X^0 - F_{01}X^1) = 0$$

If we define

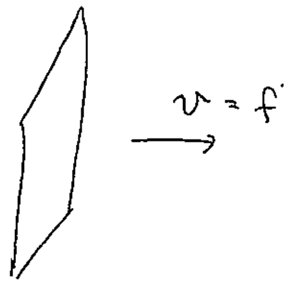
$$X^{0\prime} = \frac{X^0 - fX^1}{\cancel{\sqrt{1+f^2}}\ \sqrt{1-f^2}} \qquad X^{1\prime} = \frac{X^1 - fX^0}{\cancel{\sqrt{1+f^2}}\ \sqrt{1-f^2}}$$

this is the Lorentz boost and the B.C. ~~are~~ is

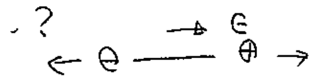
$$\partial_\tau X^{1\prime} = 0 \qquad D$$

$$\partial_\sigma (X^0)' = 0 \qquad N$$

Thus turning on ~~an~~ electric field correspond to boosting the T-dual brane. The ~~end point of~~ D-brane is moving with the constant velocity.



Note that since $|v| < 1$ there is a critical electric field in string theory.



Now turn on $F_{12} = b$ only.

we have

$$\partial_\sigma X^1 + F^{12}\partial_\tau X_2 = 0 \qquad \text{or} \qquad \partial_\sigma X^1 + b\,\partial_\tau X^2 = 0$$

$$\partial_\sigma X^2 + F^{21}\partial_\tau X_1 = 0 \qquad\qquad \partial_\sigma X^2 - b\,\partial_\tau X^1 = 0$$

T-dualize along X^1 to get the B.C.

$$\partial_\tau X^1 + b\,\partial_\tau X^2 = 0$$

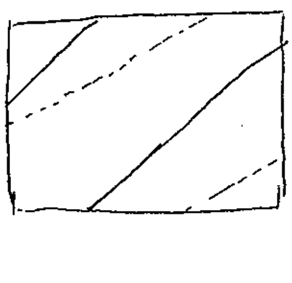
$$\partial_\sigma X^2 - b\,\partial_\sigma X^1 = 0$$

Define $\displaystyle X^{1\prime} = \frac{X^1 + bX^2}{\sqrt{1+b^2}} \qquad X^{2\prime} = \frac{X^2 - bX^1}{\sqrt{1+b^2}}$

$$\partial_\tau (X^1)' = 0 \quad D \quad \text{and} \quad \partial_\sigma (X^2)' = 0 \quad N$$

Now we have rotated brane configuration

If both T^1 and T^2 are compactified, this looks like



From this picture it is plausible that we can generate ~~for~~ curved D-branes for the general magnetic field backgrounds

We have used

$$\frac{1}{2\pi\alpha'}\int \sqrt{-h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \frac{1}{2\pi\alpha'}\int \varepsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \quad - \otimes$$

Classically equivalent action is given by

$$\frac{1}{2\pi\alpha'}\int \sqrt{-G} + \frac{1}{2\pi\alpha'}\int \varepsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \qquad \otimes\otimes$$

From the equation motion for h

$$-\frac{1}{2} h^{\alpha\beta} h_{\gamma\delta} X^{\gamma\delta} + G^{\alpha\beta} = 0 \qquad \text{we set} \qquad h_{\alpha\beta} = X_{\alpha\beta}$$

$$X_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \qquad\qquad = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$$

i.e $h_{\alpha\beta}$ is the same as the induced metric

For simplicity, we check the case with $G_{\mu\nu} = \eta_{\mu\nu}$

$B_{\mu\nu} =$ constant

the 2nd terms does not contribute to the eqn of motion since it is the total derivative.

eqn of motion from $\otimes$ for X^μ

$$\partial_\alpha (\sqrt{-h}\, h^{\alpha\beta} \partial_\beta X_\mu) = 0$$

eqn of motion from $\otimes\otimes$ — we use $\delta \det G_{\alpha\beta} = \det G \cdot G^{\alpha\beta} \delta G_{\alpha\beta}$

$$\partial_\alpha (\sqrt{-G}\, G^{\alpha\beta} \partial_\beta X_\mu) = 0$$

Thus if $h_{\alpha\beta} = G_{\alpha\beta}$, we have the same e.o.m.

For our purpose, it is more convenient to use $\otimes\otimes$, Nambu-Goto action.

Note that $\int \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}$ is the 2-D generalization of

$$\int d\tau\, \partial_\tau X^\mu A_\mu$$

Charged particles can couple to A_μ

Similarly charged strings can couple to $B_{\mu\nu}$

should be defined w.r.t $B_{\mu\nu}$

Now we are playing a curious game.
We start from the 11-D, and consider 3-dimensional extended object, called a membrane. Then compactify the 11-th circle.
We let this 3-dimensional object wrap the 11-th circle. If the 11-th circle is sufficiently small, this looks like 2-D object.
Let us take the membrane action, similar to the string action

$$S_M = T_3 \left(\int d^3\sigma \sqrt{-\det \partial_\mu X^M \partial_\nu X^N g_{MN}} + \int \varepsilon^{\mu\nu\lambda} \partial_\mu X^L \partial_\nu X^M \partial_\lambda X^N C \right.$$

Take $ds_{11}^2 = e^{-\frac{2}{3}\phi} g_{mn} dx^m dx^n + e^{-\frac{2}{3}\phi} e^{2\phi} (dx^{11} + A_m dx^m)^2$

ϕ 10-D

$$= g_{MN} dX^M dX^N$$

Note that the radius of the 11-th circle is $e^{\frac{4}{3}\phi}$

let $\mu = (\alpha, \rho)$

$X^M = (X^m, X^{11})$ and set $\rho = X^{11} \cdot l$

then

for $\alpha, \beta = 1, 2$ $\quad \partial_\alpha X^M \partial_\beta X^N g_{MN} = \partial_\alpha X^m \partial_\beta X^n e^{-\frac{2}{3}\phi} (g_{mn} + e^{2\phi} A_m A_n)$

$$\partial_\alpha X^M \partial_3 X^N g_{MN} = \partial_\alpha X^m e^{-\frac{2}{3}\phi} e^{2\phi} A_m$$

$$\partial_3 X^M \partial_3 X^N g_{MN} = g_{11\,11} = e^{\frac{4}{3}\phi}$$

$$T_3 \int d^3\sigma \sqrt{-\det \partial_\alpha X^M \partial_\beta X^N g_{MN}}$$

$$= T_3 \cdot 2\pi l \int d^2\sigma \sqrt{-\det \partial_\alpha X^m \partial_\beta X^n g_{mn}}$$

$$T_3 \int d^3\sigma \; \varepsilon^{\mu\nu\lambda} \partial_\mu X^L \partial_\nu X^M \partial_\lambda X^N C_{LMN}$$

$$= T_3 \cdot 2\pi l \int d^2\sigma \; \varepsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n C_{mn\,11}$$

Thus if we identify $C_{mn\,11} = B_{mn}$, $T_3 \cdot 2\pi l = \frac{1}{2\pi\alpha'}$

we obtain the classical string action

Starting from the membrane action and by doing double dimensional reduction, we obtained the string action with the strange relation between the 10-D metric and the 11-D metric.

Note that $R_{11}^2 = e^{\frac{4}{3}\phi}$ $\quad R_{11} = e^{\frac{2}{3}\phi}$.

e^ϕ is the usual dilaton, thus if we literally take this result, then it implies as we take the string coupling constant large enough, another dimension opens up.

This is the essence of the duality relation between 10-D string theory and the 11-D theory, which is called the M theory.

This calculation was done in 1987. But only 8 years after, the evidences for the existence of 11-D theory started accumulating.

Up to now, we have huge evidences for the M th as the strong coupling limit of 10-D string theory. About 11-D M th, only its low energy limit is known — which is called the 11-D super gravity, whose bosonic massless modes are $g_{\mu\nu}$ and $C_{\mu\nu\lambda}$.

Note that the charged particle acts as a source for the Maxwell field and we saw the coupling of the particle and the e.m. field. Similarly in the string theory, we have massless gauge fields $B_{\mu\nu}$ and we have the coupling between the string and the $B_{\mu\nu}$ field.

The string can also act as a source for $B_{\mu\nu}$.

In the 11-D th, we have $C_{\mu\nu\rho}$ and the membrane couples to it.

Membrane is also the charged source for $C_{\mu\nu\rho}$.

There is also an object which is magnetically coupled to $C_{\mu\nu\rho}$, called the 5-brane, 6-dim object.

Now dimensionally reduce the M-theory on a circle

$$g_{\mu\nu} \rightarrow g_{\mu 11} \quad g_{11\,11} \quad g_{\alpha\beta}$$

$$C_{\mu\nu\lambda} \rightarrow \underset{\text{string}}{C_{\mu\nu 11}} \quad \underset{D2}{C_{\mu\nu\lambda}}$$

(D0)

		electric	magnetic	
$g_{\mu 11}$	$\sim A_\mu$	D0 ,	D6	
$C_{\mu\nu 11}$	$\sim B_{\mu\nu}$	string	5-brane	also SUSY object
$C_{\mu\nu\rho}$	--	D2	D4	

there are also D8-branes, as well coupled to 9-form potential
10-form field strength
$\updownarrow$
cosmological constant

So far we have introduced the various extended object and these have the couplings

$$\int_{M_{p+1}} C_{p+1} = \int d^{p+1}x \cdot \varepsilon^{\alpha_1 \cdots \alpha_{p+1}} \partial_{\alpha_1} X^{M_1} \partial_{\alpha_2} X^{M_2} \cdots \partial_{\alpha_{p+1}} X^{M_{p+1}} \otimes C_{M_1 M_2 \cdots M_{p+1}}$$

Now this extended object has also the coupling like

$$\int F \wedge C_{p-1} \qquad \int F \wedge F \wedge C_{p-3} \text{ etc.}$$

Consider N Dp-branes wrapping m times on the torus T^2 and turn on n units of magnetic flux on T^2.

Thus $\quad m \cdot \frac{1}{2\pi} \int_{T^2} F = n$

A D(p-2)-brane is described by $\qquad (\partial_\sigma X^M + F^{M\nu} \partial_\tau X_\nu = 0)$
taking $(m, n) = (0, 1)$. The open string ending on a p-brane wrapped on a two-torus with magnetic flux become Dirichlet for (formally) infinite magnetic field.

This suggests that $\quad (m)\int_{M_{p+1}} F \wedge C_{p-1} \sim \int_{M_{p-1}} C_{p-1}$

The exact coupling is of the form $\quad \int e^{B + 2\pi\alpha' F} \wedge C$

Going back to the 10-D string theory, we have D0, D2, D4, D6, D8 branes which are sources for the various form fields.

If we compactify on a circle and T-dualize, we can have D1, D3, D5, D7, D9 branes.

So apparently we obtain the theory which has

$B^{(2)}_{\mu\nu} \quad A^{+}_{\mu\nu\rho\sigma} \quad \phi^{(2)}$

In fact in 10-D, we have two kinds of the superstring theory

IIA $\quad g_{\mu\nu}.\ \phi.\ B_{\mu\nu},\ A_\mu,\ C_{\mu\nu\rho}$

IIB $\quad g_{\mu\nu}.\ \underbrace{\phi^{(1)}\ B_{\mu\nu}^{(1)}}_{NS\text{-}NS},\ \phi^{(2)}.\ \underbrace{B_{\mu\nu}^{(2)}}_{R\text{-}R},\ A^{+}_{\mu\nu\rho\sigma}$

fundamental string $\qquad$ self-dual

we have two kinds of scalars $\phi^{(1)}$. $\phi^{(2)}$

two kinds of two form field $B_{\mu\nu}^{(1)}$. $B_{\mu\nu}^{(2)}$

$$\lambda = \chi + i e^{-\phi}$$

$$\lambda \to \frac{a\lambda + b}{c\lambda + d} \qquad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2\ R)$$

$$B^{(1)} \to d B^{(1)} - c B^{(2)} \qquad B^{(2)} \to a B^{(2)} - b B^{(1)}$$

F_5 is not changed

$\to$ D3-brane is $SL(2\text{-}R)$ invariant

D1 and F1 are $SL(2.R)$ pair.

$$S = -\int d^n\sigma \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} \quad - \circledast \qquad n = p+1$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Recast the theory in terms of a dual $(p-2)$-form potential $B_{\mu_1\mu_2 \ldots \mu_{p-2}}$ given by

$$-\frac{\delta S}{\delta F_{\mu\nu}} = \tilde{H}^{\mu\nu} = \frac{1}{(p-2)!} \varepsilon^{\mu\nu\lambda\rho_1 \ldots \rho_{p-2}} \partial_\lambda B_{\rho_1 \ldots \rho_{p-2}} \qquad \ast\ast$$

The Bianchi identity for the B field is the field eqn for the Maxwell field. And the Bianchi for the Maxwell provides the field eqn for the B field.

Solving for $F_{\mu\nu}$. Equivalently add a Lagrange multiplier term $\frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu)$ to $\ast$ and eliminate F.

To solve eq $\ast\ast$ for $F_{\mu\nu}$, use the Lorentz invariance to bring $F_{\mu\nu}$ to

$$F_{\mu\nu} = \begin{pmatrix} 0 & f_1 & & \\ -f_1 & 0 & & \\ & & 0 & f_2 \\ & & -f_2 & 0 \end{pmatrix}$$

Then $\ast\ast$ implies that

$$\tilde{H}^{\mu\nu} = \begin{pmatrix} 0 & h_1 & & \\ -h_1 & 0 & & \\ & & 0 & h_2 \\ & & -h_2 & 0 \end{pmatrix}$$

$$h_i = \frac{f_i}{1+f_i^2}\sqrt{\Pi(1+f_j^2)} \qquad p \leq 4$$

$$f_i = \frac{h_i}{1-h_i^2}\sqrt{\Pi(1-h_j^2)}$$

From this information, we can see that eq of motion for $\tilde{H}$ is derived from

$$S = -\int d^n\sigma \sqrt{-\det(\eta_{\mu\nu} + i\tilde{H}_{\mu\nu})}$$

If we consider more general action

$$S = -\int d^n\sigma \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + B_{\mu\nu})} \quad + \quad \frac{1}{2} F_{\mu\nu} C^{\mu\nu}$$

the equivalent action is given by

$$S = -\int d^n\sigma \sqrt{-\det(G_{\mu\nu} + i(K_{\mu\nu} + C_{\mu\nu})} \quad - \quad \frac{1}{2} A^{\mu\nu} B_{\mu\nu}$$

with $\displaystyle K_{\mu\nu} = \frac{1}{\sqrt{-\det G}} G_{\mu\rho} G_{\nu\lambda} \tilde{F}^{\rho\lambda}$

D2-brane

$$S = \int d^3\sigma \; -e^{-\phi}\sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + b_{\mu\nu})} \; + \tfrac{1}{2}\hat{A}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu)$$

$$- \int e^{-\phi}(C_3 + C_1 \wedge (F + b)) \qquad \int C_1 \wedge F = \int \tfrac{1}{2} C_1{}^{\mu\nu} F_{\mu\nu}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel \; \varepsilon^{\mu\nu\lambda\sigma} C_{1\lambda\sigma}$$

$$A_\nu \;\; \text{e.o.m} \;\; \xrightarrow{=} \;\; \hat{A}^{\mu\nu} = \varepsilon^{\mu\nu\lambda}\partial_\lambda B$$

Eliminating the U(1) gauge field in favor of the dual scalar B

$$S_D = -\int d^3\sigma \; e^{-\phi}\sqrt{-\det G'_{\mu\nu}} \; - \int e^{-\phi} C_3 + (e^{-\phi} C_1 - B) \wedge b$$

where $G'_{\mu\nu} = G_{\mu\nu} + (-e^{\phi}\partial_\mu B + \overset{C_1}{C_\mu})(-e^{\phi}\partial_\nu B + C_\nu)$

$$S_D = -\int d^3\sigma \; e^{-\phi}\sqrt{-\det G'_{\mu\nu}} \; + \int e^{-\phi}\Omega_D$$

$$\Omega_D = C_3 + (C_1 - Be^{-\phi}) \wedge b$$

The dilaton dependence can be absorbed by the rescaling

$$X^M \to e^{\frac{1}{3}\phi} X^M$$

$$S_D = -\int d^3\sigma \sqrt{-\det G''_{\mu\nu}} \; + \int \Omega''$$

$$G''_{\mu\nu} = e^{-\frac{2}{3}\phi} G_{\mu\nu} + e^{\frac{4}{3}\phi}(-\partial_\mu B + e^{-\phi}C_\mu)(-\partial_\nu B + e^{-\phi}C_\nu)$$

$$= e^{-\frac{2}{3}\phi} G'_{\mu\nu}$$

This is the relation we have seen before

$$ds_{11}{}^2 = e^{-\frac{2}{3}\phi} \overset{\eta_{mn}}{g_{mn}} dx^m dx^n + e^{-\frac{2}{3}\phi} e^{2\phi}(dx^{11} + e^{-\phi} C_m dx^m)^2$$

If we define the quantity Π^M_μ $\quad G''_{\mu\nu} = \Pi^M_\mu \Pi^N_\nu \eta_{MN}$ $\qquad B \sim X^{11}$

Then $\Pi^{11}_\mu = e^{\frac{4}{3}\phi}(\partial_\mu X^{11} + e^{-\phi} C_\mu)$ $\qquad$ for $g_{mn} = \eta_{mn}$, $\Pi^m_\mu = \partial_\mu X^m$

and $\displaystyle\int \Omega'' = \int \Pi^m_\mu \Pi^n_\nu \Pi^l_\lambda C_{mnl}$

$$+ \; \Pi^m_\mu \Pi^n_\nu \overset{(\partial_\lambda X^{11} + e^{-\phi}C_\lambda)}{\Pi^{11}_\lambda} C_{mn11}$$

D1- brane

$$S = \int d^2\sigma \ - e^{-\phi} \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + b_{\mu\nu})} \quad + e^{-\phi} \int C_2$$

we can add a total derivative term

$$S = \int d^2\sigma \ - e^{-\phi} \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + b_{\mu\nu})} \ + \ \int e^{-\phi} C_2 - C_0 F$$

$$S' = \int d^2\sigma \ - e^{-\phi} \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + b_{\mu\nu})} \ + \frac{1}{2} \tilde{\Lambda}^{\mu\nu} (\underline{F_{\mu\nu}} - 2\partial_\mu A_\nu)$$

$$+ \ \frac{1}{2} e^{-\phi} \varepsilon^{\mu\nu} C_{\mu\nu} \ - \ \frac{1}{2} C_0 \underline{\varepsilon^{\mu\nu} F_{\mu\nu}}$$

Varying A_ν gives $\quad \partial_\mu \tilde{\Lambda}^{\mu\nu} = 0 \quad \to \quad \tilde{\Lambda}^{\mu\nu} = \varepsilon^{\mu\nu} \tilde{\Lambda}$

$$S_D = \sqrt{e^{-2\phi} + (\tilde{\Lambda} - C_0)^2}$$

$$S_D = - \sqrt{e^{-2\phi} + (\tilde{\Lambda} - C_0)^2} \int d^2\sigma \sqrt{-\det G_{\mu\nu}}$$

$$+ \quad \int e^{-\phi} C \ - \int \tilde{\Lambda} \, b_{\mu\nu}$$

the tension is modified $\to$ exhibit tension formula

with $\sqrt{e^{-2\phi} + (\tilde{\Lambda} - C_0)^2}$

if $\Lambda = m$ this is the $(m, 1)$ string with constant dilaton ϕ C_0

or fundamental string with an $SL(2,\mathbb{Z})$ transf. metric, dilaton and axion

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\Lambda \end{pmatrix}$$

$$C_0 + i e^{-\phi} \ \to \ \frac{-1}{C_0 + i e^{-\phi} - \Lambda}$$

$$C_0 + i e^{-\phi} \ \to \ \frac{1}{C_0 - \Lambda + i e^{-\phi}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\Lambda \end{pmatrix}$$

$$B^{(1)} = d B^{(1)} - c B^{(2)}$$

$$= -\tilde{\Lambda} B^{(1)} + B^{(2)}$$

The coupling constant after the duality transf.

$$e^{\tilde{\phi}} = e^{-\phi} + e^{\phi} (\Lambda - C_0)^2$$

($(q_1 \, q_2)$ string has $T_q = \sqrt{\Delta_q} \, T$

$\Delta_q = \quad e^{\phi_0} (q_1 \ldots$

D3- brane

$$S = -\int d^4\sigma\, e^{-\phi}\sqrt{-\det(G_{\mu\nu}+F_{\mu\nu}+b_{\mu\nu})} \quad + \quad \int e^{-\phi}\,(C_4 + C_2\wedge(F+b))$$

To set the Einstein metric, which is invariant under $SL(2,\mathbb{R})$

$$X^m \to e^{\frac{\phi}{4}} X^m$$

$$S = -\int d^4\sigma \sqrt{-\det\left(G_{\mu\nu} + e^{-\frac{\phi}{2}}(F_{\mu\nu}+b_{\mu\nu})\right)} \quad + \int (C_4 + C_2\wedge(e^{-\frac{\phi}{2}})(F+b))$$

$$+ \quad \frac{1}{2}\tilde{F}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu) \quad + \quad \frac{1}{2}C_0\, F\wedge F$$

After the dual transformation we get

$$S' = -\int d^4\sigma \sqrt{-\det\left(G_{\mu\nu} + e^{-\frac{\phi'}{2}}(\tilde{F}_{\mu\nu} + \tilde{B}_{\mu\nu})\right)}$$

$$+ \int (C_4 + \tilde{C}_{2\mu\nu}\wedge e^{-\frac{\phi'}{2}}(\tilde{F}_{\mu\nu} + \tilde{B}_{\mu\nu}) \quad + \frac{1}{2}C_0'\,\tilde{F}\wedge\tilde{F}$$

where
$$e^{\phi'} = \frac{1}{e^{\phi} + e^{+\phi}C_0^2}$$

$$C_0' = -\frac{e^{\phi}C_0}{e^{-\phi} + e^{\phi}C_0^2}$$

$$\tilde{B}_{\mu\nu} = C_{\mu\nu} \qquad \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\tilde{C}_{\mu\nu} = -B_{\mu\nu}$$

cf) The relation between the string frame and the Einstein frame

$$\int \sqrt{-g}\, e^{-2\Phi}\left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) d^Dx$$

$$\hat{g}_{ab} = \Omega^2 g_{ab}$$

$$\hat{R} = \Omega^{-2}\left(R - 2(D-1)\,g^{ac}\nabla_a\nabla_c \ln\Omega - (D-2)(D-1)\,g^{ac}\nabla_a\ln\Omega\,\nabla_c\ln\Omega\right)$$

We set $\Omega = e^{\frac{-2}{D-2}\Phi}$

$$= \int \sqrt{-\hat{g}}\left(\hat{R} - 4\,\frac{1}{D-2}(\nabla\Phi)^2 - \frac{1}{12}e^{-\frac{8}{D-2}\Phi}H^2\right) d^Dx$$